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2000 J. Phys. A: Math. Gen. 33 4073

(http://iopscience.iop.org/0305-4470/33/22/310)

J. Phys. A: Math. Gen. **33** (2000) 4073–4079. Printed in the UK PII: S0305-4470(00)09117-4

The Higgs algebra and the Kepler problem in R^3

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Received 1 November 1999, in final form 11 April 2000

Abstract. The quantum mechanical correspondence of the Kepler problem in R^3 and the free-particle motion on spaces S^3 and S_1^3 is found by using the fact that the Higgs algebra is a finite W-algebra obtained by embedding of algebra $sl(2)$ into $sl(4)$.

1. Introduction

The Kepler problem in flat spaces has been intensively studied from the symmetry point of view. The correspondence between this problem in R^n and free motion on the sphere S^n had been established in [1–3] on both classical and quantum levels. At the same time the symmetry properties of the Kepler problem in the spaces of constant curvature have been studied considerably less. In this paper we consider the quantum mechanical correspondence between the Kepler problem in R^3 and free motion on the sphere S^3 on the basis of the relations between symmetry algebras of both problems.

The symmetry algebra of the quantum mechanical Kepler problem on spaces of constant curvature has been studied in [4, 5] (for arbitrary dimensions) and independently in [6, 7] (for spaces S^3 and S_1^3). In spaces S^3 and S_1^3 the operators of angular momentum $L_i = -\mathrm{i}\varepsilon_{ijk}\xi_j\partial/\partial\xi_k$, $(i, j, k = 1, 2, 3)$ and operators of Runge–Lentz vector $\underline{R} = (1/2\rho)\{\underline{L} \times \underline{N} - \underline{N} \times \underline{L}\} + \mu\underline{\xi}/|\underline{\xi}|$ (where $\underline{N} = -\mathrm{i}(\xi_4\partial/\partial\underline{\xi} - \underline{\xi}\partial/\partial\xi_4)$) commute with the Hamiltonian on the sphere $K = -(1/4\rho^2)M_{\alpha\beta}M_{\alpha\beta} - (\mu/\rho)\xi_4/|\underline{\xi}|$ ($\underline{\xi} = \{\xi_1, \xi_2, \xi_3\}$, $\underline{\xi}^2 + \xi_4^2 = \rho^2$), where $M_{\alpha\beta} = (\varepsilon_{ijk}L_k, \varepsilon_{ijk}N_k)$ are generators of the geometrical $SO(4)$ group on a sphere. The operators R_i and L_i satisfy the following commutational relations:

$$[L_i, L_j] = \mathrm{i}\varepsilon_{ijk}L_k \qquad [L_i, R_j] = \mathrm{i}\varepsilon_{ijk}R_k \qquad [R_i, R_j] = -2\mathrm{i}\left(H - \frac{\underline{L}^2}{\rho^2}\right)\varepsilon_{ijk}L_k. \tag{1}$$

Moreover, $\boldsymbol{RL} = \boldsymbol{LR} = 0$ and $\boldsymbol{R}^2 = 2H(\boldsymbol{L}^2 + 1) - 1/\rho^2\boldsymbol{L}^2(\boldsymbol{L}^2 + 2) + \mu^2$. For spaces S_1^3 one needs a substitution: $\rho \to \mathrm{i}r$, $(r > 0)$ and $\xi_4 \to \mathrm{i}\xi_0$. It was pointed out in [9] that it is a finite W-algebra obtained by embedding $sl(2) \to sl(4)$. Finite W-algebras have been introduced [8,9] by considering symplectic reductions of finite-dimensional simple Lie algebras in complete analogy with usual (infinite-dimensional) W-algebras constructed as reductions of affine Lie algebras.

Based on the property of finite W-algebras to appear as a commutant of a particular subalgebra in a simple Lie algebra $\mathcal{G}$ in [11, 12] a new class of $\mathcal{G}$ representations has been constructed with the use of finite W-algebra associated with the above-mentioned embedding.

In this paper we proceed in a somewhat opposite manner: starting from the particular representation of $\mathcal{G}$ (we consider the $sl(4)$ case only) as a dynamical symmetry algebra of the Kepler problem in R^3 we construct a finite W-algebra in terms of this representation. Then one can associate the W generators expressed in terms of this particular representation as a Runge–Lentz vector of some problem on a sphere (section 3). Within our purely algebraic approach the natural way to express the sphere coordinates through coordinates $(\boldsymbol{x}, \boldsymbol{p})$ of the Kepler problem is to consider some orbits in coadjoint representation of $SO(4, 2)$. The explicit expressions of these coordinates (see section 4) are given by some limitations of the corresponding moment map. Therefore, one can find the correspondence between the problem defined in flat space (and possessing an algebra of symmetry $so(4, 2)$) and the problem defined on a space of constant curvature.

Some information concerning the technique of $sl(2)$ embedding is given in section 2 (for more details see [8–12]).

Throughout the paper we will use following notations for the Hamiltonians: K is the Hamiltonian of free motion on the sphere S^3 and H is the Hamiltonian of the Kepler problem in R^3.

2. General construction of *sl*(2) embedding

Let $\mathcal{G}$ be a real, connected, noncompact Lie algebra. Let $\{t_a\}$ be the basis of $\mathcal{G}$, and J^a the basis in dual space $\mathcal{G}^*$:

$$[t_a, t_b] = f_{ab}^c t_c \qquad J^a(t_b) = \delta_b^a. \tag{2}$$

Here the metric on $\mathcal{G}$ in a representation R is

$$\eta_{ab} = \langle t_a, t_b \rangle = \mathrm{tr}_R(t_a t_b) \qquad \eta_{ab}\eta^{bc} = \delta_a^c. \tag{3}$$

One can define on $\mathcal{G}^*$ a Poisson–Kirillov structure which mimics the commutators (2):

$$\{J^a, J^b\} = f_c^{ab} J^c \qquad f_c^{ab} = \eta^{ad}\eta^{be}\eta_{cg} f_{de}^g. \tag{4}$$

As usual, we introduce a gradation on $\mathcal{G} = \mathcal{G}_- \oplus \mathcal{G}_0 \oplus \mathcal{G}_+$ relative to an $sl(2.R)$ embedding into algebra $\mathcal{G}$ [9]. Let t_0, t_+, t_- form an $sl(2)$ subalgebra of $\mathcal{G}$ ($[t_0, t_\pm] = \pm 2t_\pm$, $[t_+, t_-] = t_0$). Then, its Cartan generator t_0 defines the gradation:

$$\mathcal{G} = \oplus_{p=-m}^{m} \mathcal{G}_p \qquad [t_0, X] = pX \qquad \forall X \in \mathcal{G}_p \tag{5}$$

where $[\mathcal{G}_p, \mathcal{G}_q] \subset \mathcal{G}_{p+q}$ and p are integers here.

In the usual Hamiltonian approach [9] we impose a first class constraint on the $\mathcal{G}_+$ part of the J matrices. These constraints generate a gauge invariance. Therefore, one can find gauge-independent quantities. In general, this set will be generated by some finite subset. The Poisson brackets between them form a finite W-algebra.

We introduce on $\mathcal{G}^*$ first class constraints relative to the $\mathcal{G}_+^*$ part of the J-matrix:

$$J^i - \chi^i = 0 \qquad \forall i | t_i \in \mathcal{G}_+ \tag{6}$$

where χ^i is a constant, which is zero except when $J^i = J^+$, a positive $sl(2)$ root generator. For simplicity $\chi^+ = 1$. The constraints weakly commute among themselves and generate a gauge invariance on the J^a:

$$J^a \to \exp(c_i\{J^i, .\}_{\mathrm{Const}})(J^a) = J^a + c_i\{J^i, J^a\}_{\mathrm{Const}} + \tfrac{1}{2}c_i c_j\{J^j, \{J^i, J^a\}\}_{\mathrm{Const}} + \cdots \tag{7}$$

where $J^i \in \mathcal{G}_+^*$ and c_i are the gauge transformation parameters.

Let us introduce the constrained matrix $J = t_+ + J^\beta t_\beta + J^{\bar\beta} t_{\bar\beta}$ ($J^\beta \in \mathcal{G}_0^*$ and $J^{\bar\beta} \in \mathcal{G}_-^*$) and look at the gauge transformation (7): $J \to J^g = \exp(c_i\{J^i, .\}_{\mathrm{Const}})(J)$. Developing J^g with the help of the gradation we can rewrite J^g as $J^g = g^{-1}Jg$, where $g = \exp(c^{\bar{\beta}}t_{\bar{\beta}}) \in G_-$. One fixes the gauge by demanding the transformed current J^g to be of the form

$$J^g = t_+ + W^s l_s \qquad [t_-, l_s] = 0 \qquad [t_0, l_s] = sl_s. \tag{8}$$

W-generators are in the enveloping algebra of $\mathcal{G}^*$ and are gauge invariant.

3. The Kepler problem in R^3 and free motion on the sphere S^3

It is well known that a Kepler problem in R^3 with Hamiltonian $H = p^2/2m - \alpha/r$ (where $r = (\boldsymbol{x}\boldsymbol{x})^{1/2}$, $\boldsymbol{x} = (x_1, x_2, x_3)$) has the $SO(4, 2)$ group of dynamical symmetry (see [13, 14]).

Now we can take a particular realization of $SO(4, 2)$, namely the algebra of dynamical symmetry of the Kepler problem, in order to construct a new realization of a finite W-algebra related to embedding into $sl(4)$. Both algebras $sl(4)$ and $so(4, 2)$ are noncompact versions of $so(6)$ and therefore could be connected through a compactification process.

The vector of angular momentum $\boldsymbol{J}$ together with Runge–Lentz vector $\boldsymbol{A}$ forms the $so(4)$ compact subalgebra. The classical expressions for generators of the full algebra are

$$\begin{aligned}
&\boldsymbol{J} = \boldsymbol{x} \times \boldsymbol{p} \\
&\boldsymbol{A} = \frac{1}{\sqrt{-2mH}}\left(\boldsymbol{p} \times \boldsymbol{J} - \frac{\alpha\boldsymbol{x}}{r}\right) \\
&\boldsymbol{M} = r\boldsymbol{p}\cos\delta + \frac{1}{\sqrt{-2mH}}\left((\boldsymbol{x}\boldsymbol{p})\boldsymbol{p} - \frac{m\alpha\boldsymbol{x}}{r}\right)\sin\delta \\
&\boldsymbol{\Gamma} = -r\boldsymbol{p}\sin\delta + \frac{1}{\sqrt{-2mH}}\left((\boldsymbol{x}\boldsymbol{p})\boldsymbol{p} - \frac{m\alpha\boldsymbol{x}}{r}\right)\cos\delta \\
&\Gamma_0 = \frac{m\alpha}{\sqrt{-2mH}} \\
&T = \frac{1}{\sqrt{-2mH}}(rp^2 - m\alpha)\cos\delta - (\boldsymbol{x}\boldsymbol{p})\sin\delta \\
&\Gamma_4 = -\frac{1}{\sqrt{-2mH}}(rp^2 - m\alpha)\sin\delta - (\boldsymbol{x}\boldsymbol{p})\cos\delta
\end{aligned} \tag{9}$$

where $\delta = -(\boldsymbol{x}\boldsymbol{p})/\Gamma_0$. Note also that $\boldsymbol{J}\boldsymbol{A} = 0$ and $\boldsymbol{A}^2 = -\boldsymbol{J}^2 - \alpha^2/2mH$. Thus the second Casimir operator of $SO(4)$ is $C_2 = \boldsymbol{J}^2 + \boldsymbol{A}^2 = -\alpha^2/2H$.

According to the general procedure described in section 2 we identify these quantities with the coordinates of points in $so(4, 2)^*$. For embedding $sl(2) \to sl(4)$ we choose the following basis in $sl(4)$:

$$J^a t_a = \begin{pmatrix}
\frac{1}{2}J_{\Gamma_0} + J_{j_3} + J_{A_3} & J_{j_1+\mathrm{i}j_2} + J_{A_1+\mathrm{i}A_2} & J_{\Gamma_4-\mathrm{i}T} + J_{\Gamma_3-\mathrm{i}M_3} & J_{\Gamma_1-\mathrm{i}M_1+\Gamma_2-\mathrm{i}M_2} \\
J_{j_1-\mathrm{i}j_2} + J_{A_1-\mathrm{i}A_2} & \frac{1}{2}J_{\Gamma_0} - J_{j_3} - J_{A_3} & J_{\Gamma_1-\mathrm{i}M_1-(\Gamma_2-\mathrm{i}M_2)} & J_{\Gamma_4-\mathrm{i}T} - J_{\Gamma_3-\mathrm{i}M_3} \\
J_{\Gamma_4+\mathrm{i}T} + J_{\Gamma_3+\mathrm{i}M_3} & J_{\Gamma_1+\mathrm{i}M_1+\Gamma_2+\mathrm{i}M_2} & -\frac{1}{2}J_{\Gamma_0} + J_{j_3} - J_{A_3} & J_{j_1+\mathrm{i}j_2} - J_{A_1+\mathrm{i}A_2} \\
J_{\Gamma_1+\mathrm{i}M_1-(\Gamma_2+\mathrm{i}M_2)} & J_{\Gamma_4+\mathrm{i}T} - J_{\Gamma_3+\mathrm{i}M_3} & J_{j_1-\mathrm{i}j_2} - J_{A_1-\mathrm{i}A_2} & -\frac{1}{2}J_{\Gamma_0} - J_{j_3} + J_{A_3}
\end{pmatrix}.$$

Here the quantities J_{Γ_0}, J_{j_1} etc denote the elements in the space $so(4, 2)^*$ which corresponds to the elements of $so(4, 2)$ in the sense of metric (2). The $sl(2)$ subalgebra we consider is $t_0 = t_{\Gamma_0}$, $t_+ = t_{\Gamma_4+\mathrm{i}T}$, $t_- = t_{\Gamma_4-\mathrm{i}T}$. The corresponding grading of $sl(4) = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1$ is the following:

$$\begin{aligned}
&\mathcal{G}_{-1} = \{\Gamma_4 - \mathrm{i}T, \Gamma_k - \mathrm{i}M_k\} \\
&\mathcal{G}_0 = \{\boldsymbol{J}_i, \Gamma_0, \boldsymbol{A}_k\} \\
&\mathcal{G}_1 = \{\Gamma_4 + \mathrm{i}T, \Gamma_k + \mathrm{i}M_k\}.
\end{aligned} \tag{10}$$

The one-dimensional representation is defined as $\chi(t_+) = 1$, $\chi^i(t_{\Gamma_k+\mathrm{i}M_k}) = 0$. The constraints therefore read $J^+ - 1 = 0$, $J^{\Gamma_k+\mathrm{i}M_k} = 0$.

One has to find gauge independent quantities as functions of variables J_k, Γ_0, A_k. The corresponding group element G_+ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a+b & c & 1 & 0 \\ d & a-b & 0 & 1 \end{pmatrix} \tag{11}$$

where parameters a, b, c, d are functions of currents. After calculations one obtains the following expressions for W-generators:

$$\begin{aligned} \tilde{R}_1 &= \Gamma_1 + \mathrm{i}M_1 + 2\Gamma_0 A_1 + 2(J_2 A_3 - A_2 J_3) \\ \tilde{R}_2 &= \Gamma_2 + \mathrm{i}M_2 + 2\Gamma_0 A_2 + 2(A_3 J_1 - J_3 A_1) \\ \tilde{R}_3 &= \Gamma_3 + \mathrm{i}M_3 + 2\Gamma_0 A_3 + 2(J_1 A_2 - A_1 J_2) \\ C' &= \Gamma_4 - \mathrm{i}T + J_3 + \Gamma_0^2 + \boldsymbol{A}^2 + \boldsymbol{J}^2 \\ L_i &= J_i. \end{aligned}$$

These generators form the Higgs algebra (1). Now we perform the classical Miura transformation defined as follows (for more details see [9]); note that the W generators could be presented in the following way:

$$\tilde{R}^a = R_-^a + \tilde{R}_0^a \tag{12}$$

where the R_-^a part contains generators from $\mathcal{G}_-$ only, and the $\tilde{R}_0^a$ part is constructed from $so(4,2)$ generators which belong to $\mathcal{G}_0$. Therefore, one can define the embedding of a finite W-algebra expressed in terms of $\mathcal{G}_- \oplus \mathcal{G}_0$ into Kirillov–Poisson algebra $\mathcal{K}_0$ of subalgebra $\mathcal{G}_0$:

$$\mathcal{W} \longrightarrow \mathcal{K}_0(\mathcal{G}_0). \tag{13}$$

This means that the algebra generated by $\tilde{R}^a$ is isomorphic to the algebra generated by R_0^a.

Moreover one can note that it is isomorphic to the algebra with W generators constructed from the $sl(2) \oplus sl(2)$ subalgebra of $\mathcal{G}_0$. Therefore, we obtain that the generators ($a = 1, 2, 3$)

$$R_a = \mathrm{i}(J \times A)_a \qquad J_a \tag{14}$$

do form the Higgs algebra under Poisson brackets:

$$\begin{aligned} &\{R_a, R_b\} = -\mathrm{i}(C_2 - 2J^2)\varepsilon_{abc} J_c \\ &\{J_a, R_b\} = \mathrm{i}\varepsilon_{abc} R_c \qquad \{J_a, J_b\} = \mathrm{i}\varepsilon_{abc} J_c \end{aligned}$$

where $C_2 = J^2 + A^2$ is the second Casimir operator of $o(4)$.

4. The explicit construction of a sphere

Now we consider the vector $\boldsymbol{R}$ as a Runge–Lentz vector of the classical problem on some sphere. The operator C_2 (Poisson) commutes with all R_is and therefore one can interpret $C_2/2$ as a Hamiltonian of a *new problem on the sphere* (see (1)). The vectors $\boldsymbol{J}, \boldsymbol{R}$ satisfy all additional conditions on vectors of the Kepler problem in space S^3 (see section 1). The classical expressions are

$$\boldsymbol{R} \cdot \boldsymbol{J} = 0 \qquad \boldsymbol{R}^2 = \boldsymbol{J}^2 \boldsymbol{A}^2 = -\boldsymbol{J}^4 + 2K\boldsymbol{J}^2 \tag{15}$$

from which it follows that the charge $\mu = 0$. Therefore, for the classical Hamiltonian of a free particle on a sphere we have $2K = -\alpha^2/(2H)$.

The quantum counterparts of the above expressions are

$$\begin{aligned}
&R_a = \frac{\mathrm{i}}{2}(\boldsymbol{J}\times\boldsymbol{A} - \boldsymbol{A}\times\boldsymbol{J})_a \qquad J_a \\
&[R_i, R_j] = -\mathrm{i}\varepsilon_{ijk}(C_2 - 2\boldsymbol{J}^2)J_k \qquad [J_i, R_j] = \mathrm{i}\varepsilon_{ijk}R_k \qquad [J_i, J_j] = \mathrm{i}\varepsilon_{ijk}J_k \\
&\boldsymbol{J}\boldsymbol{R} = \boldsymbol{R}\boldsymbol{J} = 0 \qquad \boldsymbol{R}^2 = C_2\boldsymbol{J}^2 - \boldsymbol{J}^2(\boldsymbol{J}^2+2)
\end{aligned} \tag{16}$$

where C_2 is the quantum counterpart of the second Casimir operator. Note that the expression for operator $\boldsymbol{A}$ is valid for $H = E_n = -\alpha^2/(2(\boldsymbol{J}^2+\boldsymbol{A}^2+1)) < 0$. Taking into account the quantum expression for $\boldsymbol{A}^2 = -(\boldsymbol{J}^2+1) - \alpha^2/(2H)$ we obtain the spectrum of the problem on a sphere:

$$2K = C_2 = -\frac{\alpha^2}{2H} - 1. \tag{17}$$

The irreducible representations of $SO(4)$ are defined by two numbers: j_1 and j_2 ($j_i \in Z_+/2$). For the representation realized in the Kepler problem $j_1 = j_2 = j$ and $C_2 = 4j(j+1)$. From the other side $j = (n-1)/2$, where n is a hydrogen quantum number. Therefore, we obtain the spectrum of free-particle motion on a unit sphere: $\epsilon_n = C_2/2 = (n^2-1)/2$, where $n = 1, 2, 3, \ldots$ is a quantum number of the hydrogen atom in R^3.

In order to consider the case $H > 0$ we note that vectors $\boldsymbol{J}, \boldsymbol{A}$ satisfy relations of the Lorentz group and thus we will have a 'plus' sign in the nonlinear term of the commutation relations for R_i and $C_2 = J^2 - A^2$. Therefore, the case $H > 0$ corresponds to the motion of a particle on the unit hyperboloid S_1^3. For the motion of a particle in the space S_1^3 in a Coulomb potential there are both discrete and continuous parts of the spectrum, but because $\mu = 0$ in our case only the continuous part exists.

In order to identify the sphere on which this motion takes place we have to obtain the expressions for $\boldsymbol{J}, \boldsymbol{A}$ in a form similar to the form of generators of $SO(4)$ on a unit sphere. Therefore, in some coordinates ξ_α generators $\boldsymbol{J}, \boldsymbol{A}$ have to be presented as follows: $M_{\alpha,\beta} = -\mathrm{i}(\xi_\alpha\partial_\beta - \xi_\beta\partial_\alpha)$ (see section 1).

In order to obtain such a representation of $so(4)$ (and therefore the reps of $so(4,2)$) it is natural to use the method of orbits.

Note that the stability subgroup of the point $\omega_l = l\Gamma_0^* \in so(4,2)^*$ is given by the maximal compact subgroup $SO(2)\otimes SO(4)$. Therefore, the orbit O_{ω_l} through this point is the homogeneous symplectic manifold $SO(4,2)/SO(2)\otimes SO(4)$. One special parametrization of this coset space is to consider it as a bounded subdomain of type IV in $\boldsymbol{C}^4$ (see, for example, [16]) that obeys the conditions

$$\xi_\alpha\bar{\xi}_\alpha < 1 \qquad 1 - 2\xi_\alpha\bar{\xi}_\alpha + |\xi_\alpha\xi_\alpha|^2 > 0 \tag{18}$$

where $\xi_\alpha (\alpha = 1\ldots 4)$ ($\xi_\alpha\xi_\alpha = \xi_1^2 + \cdots + \xi_4^2$) are complex coordinates in the domain. In this parametrization the coset space possesses a standard Kähler potential $K = -\log(1 - 2\xi_\alpha\bar{\xi}_\alpha + |\xi_\alpha\xi_\alpha|^2)$. The moment map $O_{\omega_l} \to \boldsymbol{C}^4$ is defined as follows:

$$\xi_\alpha = \frac{\mathrm{i}}{2\Gamma_0}\left(\sigma_\alpha + \frac{\sigma_\alpha\sigma_\alpha}{2\Gamma_0(\Gamma_0+l) - \sigma_\alpha\bar{\sigma}_\alpha}\bar{\sigma}_\alpha\right) \tag{19}$$

where $\sigma_\alpha = M_\alpha - \mathrm{i}\Gamma_\alpha$, $\Gamma_\alpha = (\boldsymbol{\Gamma}, \Gamma_4)$ and $M_\alpha = (\boldsymbol{M}, T)$. The action of $SO(4,2)$ on this domain is given by holomorphic transformations. In [17] the following representation of algebra $so(4,2)$ has been obtained by holomorphic induction from character $\exp[\mathrm{i}(l+1)\Gamma_0^*]$:

$$\begin{aligned}
&\hat{\boldsymbol{M}}_{\alpha\beta} = -\mathrm{i}(\xi_\alpha\partial_\beta - \xi_\beta\partial_\alpha) \qquad \hat{\boldsymbol{\Gamma}}_0 = (l+1) + \xi_\alpha\partial_\alpha \\
&\hat{\boldsymbol{\Gamma}}_\alpha = -(l+1)\xi_\alpha - \left(\xi_\alpha\xi_\beta + \frac{1-\xi_\alpha\xi_\alpha}{2}\delta_{\alpha\beta}\right)\partial_\beta \\
&\hat{\boldsymbol{M}}_\alpha = \mathrm{i}(l+1)\xi_\alpha + \mathrm{i}\left(\xi_\alpha\xi_\beta - \frac{1+\xi_\alpha\xi_\alpha}{2}\delta_{\alpha\beta}\right)\partial_\beta.
\end{aligned}$$

For $l = 0$ and real ξ_α this representation is identical with that obtained in [18]. The limit $l \to 0$ and $\sigma_\alpha\sigma_\alpha = 0$ which is realized in the Kepler problem is well defined [17]. Therefore, taking real and imaginary parts of vector ξ_α we obtain the coordinates of the spheres (in coordinate and momentum spaces respectively) under investigation ($\delta = 0$ here):

$$\left(\frac{\boldsymbol{x}}{r} - \frac{(\boldsymbol{xp})}{m\alpha}\boldsymbol{p}, \sqrt{-2mH}\frac{(\boldsymbol{xp})}{m\alpha}\right) \qquad \left(\sqrt{-2mH}\frac{r\boldsymbol{p}}{m\alpha}, \frac{rp^2}{m\alpha} - 1\right). \tag{20}$$

Thus the free-particle motion on these spaces corresponds to the Kepler motion of a particle in R^3. These coordinates are orthogonal to each other because $M_\mu \Gamma_\mu = 0$. Note that the second coordinate system has been found by Fock [19].

The coordinates for the space S_1^3 can be obtained by the substitution $\xi_4 \to \mathrm{i}\xi_4$.

In these coordinates the wavefunction of the Kepler motion in R^3 for $H < 0$ is the same as for the free motion on a sphere and the wavefunction for the continuous Kepler spectrum coincides with the wavefunction of free motion on S_1^3.

5. Conclusion

It is necessary to note the following items:

(1) The group of the dynamical symmetry of the Kepler problem in R^{D-1} is $SO(D, 2)$. The generators of the Higgs algebra can be constructed in the same way as for $D = 3$ and therefore the approach offered here can be generalized on any D.

(2) The method of construction of the representations of the Higgs algebra starting from a realization of $so(4, 2)$ as the algebra of group of dynamical symmetry can be applied to another physical system, namely a free massless relativistic particle in four-dimensional space–time.

(3) The construction allows a generalization for the case when there is a gauge connection on sphere S^{D-1} induced by embedding of this sphere into R^D according to the approach of [20].

(4) The expressions for $SO(4, 2)$ generators can be replaced by those given by Barut and Bornzin in [15]. Their realization gives quantum expressions for $SO(4, 2)$ generators. The expressions for sphere coordinates will be different from those given here.

Acknowledgments

We would like to thank Professor C Grosche for the information that stimulated this investigation, Dr V S Otchik for his useful comments and Dr I Kanatchikov for sending us a manuscript [11]. It is a pleasure to thank our colleagues Professor E A Tolkachev and Dr M Pletyukhov for assistance. We are very grateful to Professor I Mladenov for sending us a manuscript [3]. We would also like to thank our referees for much useful advice and for calling our attention to [1–3] and [14].

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